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Fundamentals of Soft Logic

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In this paper, we develop a new theory termed Soft Logic. This theory addresses the need to combine real processes and cognitive ones in the same framework. At the same time, we develop a new concept of modeling and dealing with uncertainty: the uncertainty of time and space. We develop a language that can talk in two reference frames, and also suggest a way to combine them. This constitutes a fundamental new mathematics. In this paper, we present the theory from its foundation.

In Soft Logic, we refer to situations of uncertainty occurring in the present. Ambiguity is created from the possible point of view of the observer of the phenomenon. Sometimes reality contains contradictory or opposite situations at the same time, which can be interpreted in different ways by the observer. Such is the case of the Necker cube or the Fechner color effect, in which a texture of black and white colors moving in a certain pattern creates a rainbow of colors. Another example is the Mobius strip, which from a local point of view seems to have two sides, but from a global point of view has only one.

The mathematical solution for expressing uncertainty in Soft Logic is the blow-up of the number zero by distinguishing between multiples of this number. In this manner, we create a zero-axis and define a new type of number called Soft Numbers, which have the following form:

$$a\overline{0} + b\overline{1}.$$

The zero-axis expresses the inner world of the observer and is the component of a subjective interpretation and a certain type of uncertainty that occurs in the present.

Soft Numbers are similar to complex numbers, but the specific mathematical definitions are different and therefore Soft Numbers are a different world. This paper presents five axioms of Soft Logic, which form a new system of coordinates that differs from the Cartesian system and Gauss's complex plane. We define the algebra of Soft Numbers, show the connection to differential and integral calculus, expand analytical functions to soft functions, and describe continuous curves in the soft coordinate system. In conclusion, we suggest several avenues for theoretical and practical research.

Keywords: Soft Logic; Soft Numbers; cognitive modeling; economic behavioral modeling.

1. Introduction

In this paper, we develop from its foundation a new theory termed Soft Logic. We develop a new concept of modeling and dealing with uncertainty: the uncertainty of

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time and space. We develop two reference frames with different paradigms, and then suggest a way to combine them. This constitutes a fundamental new mathematics, with many potential applications. Applications of this theory are in the combination of cognitive and physical worlds (the two axis) and in related theories (such as Economics, where internal behavioral and external factors are combined in the decision making process).

We live in a dynamic reality that is changing at an increasingly swift pace. One of the manifestations of this reality is the integration of the real world and the digital world, with each having its own rules. Children born into the world today actually live in a dual, confusing reality. This requires the development of a mathematical language that can bridge the two worlds, which ostensibly run parallel to each other, but are in fact related and affect each other.

One geometric model (that result from our theory), which is suitable for describing the mutual influence of the two different worlds, is the Mobius strip. The real world is on one side of the strip while the world of information is on the other. Locally, the strip has two sides, but globally it has only one. The new mathematical language must have a different logic that encompasses opposites and paradoxes.

The mathematician Felix Klein developed a geometric-algebraic model called "Blow-up".[1] By blowing up the point of origin of some coordinate system (Cartesian and the one defined here), a Möbius strip is formed. We add to the point of origin all the possible revolutions around it, in 360 degrees. A circle is thus created around the point of origin, which is connected by lines to each point on this circle. After turning the circle by 180 degrees, it returns on itself but in the opposite direction. When this procedure is conducted in three dimensions, reversing direction upon itself by 180 degrees creates the Möbius strip twist.

In Soft Logic we blow up the zero point by distinguishing between the various multiplications of zero and creating a zero-axis. Using five axioms, we define Soft Numbers and their algebra, which almost creates a field (proof in the paper). In this paper, we describe the soft coordinate system, the algebra of Soft Numbers, the relationship with differential and integral calculus, the expansion of soft functions and their presentation on Soft curves. To conclude we present several practical avenues of future research for Soft Logic.

1.1. *Research background*

This research is inspired by Dascal,[2] who wrote about Gottfried Wilhelm Leibniz. As a young researcher, Leibniz aspired to discover and develop a mathematical language that would demonstrate a softer logic, which would overcome the limitations of the dichotomy of true and false. Soft Logic fulfils this vision by extending the number 0 from a singular point to a continuous line, which we term the zero line.

Another project of interest to Leibniz was to develop the world of infinitesimals. He saw them as ideal numbers that might be infinitely small. Dual numbers (see

Appendix), which were presented by Clifford,[3–5] include some infinitesimals. These numbers have the following form: $\boldsymbol{a + b\varepsilon}$, where a, b are real numbers and $\boldsymbol{\varepsilon^2 = 0.}$

In the Soft Logic research[6–10] we present a new kind of number, which we call "Soft Numbers" that are of the following form:

$$a\overline{0}{+}b\overline{1}.$$

The main difference between $\boldsymbol{\varepsilon}$ and $\overline{\mathbf{0}}$ is the geometrical interpretation of $\overline{\mathbf{0}}$ as an extension of the number 0 on a continuous line. This line can be a model of the inner world, while $\overline{\mathbf{1}}$ is a model of the real world. The bridge between them enables us to treat the concept of consciousness with mathematical tools. Another difference is the possibility, in Soft Logic, of developing a Soft curve, as described in Sec. 6.

The work develops arithmetic and algebraic operations on Soft Numbers, and describes the relationships of Soft Numbers to various topics, such as calculus, and the properties and graphic presentation of soft functions. Soft Numbers can be a model of a bridge between the internal world (pure information, unmeasurable but influential) and the external world, where there is a meaning to the order of the processes, and hence it is not commutative.

1.2. *Paper organization*

Section 2 presents the five axioms of Soft Logic. Based on our previous paper,[8] we added two more axioms: the bridge axiom and the axiom of non-commutativity. Section 3 presents in brief form our new coordinate system of Soft Logic. Section 4 deals with the algebra of Soft Numbers. Section 5 states the connection between Soft Numbers and calculus. Section 6 shows how to extend a real function to a soft function and to present it graphically in the Soft Numbers coordinate system. Section 7 contains conclusions and perspectives.

2. Axioms and Definitions

According to traditional mathematics, the expression $0/0$ is undefined, although in fact any real number could represent this expression, since $a * 0 = 0$ for all real numbers a. This observation opens a new scope and area for investigation.

We assume an existence of a continuum of distinct multiples $a\overline{0}$, where a is any real number and by $\overline{0}$ we symbolize some object that may be called a 'soft zero', while its multiples are called soft zeros as well. We denote by $\overline{1}$ the real number 1 when all other real numbers are conceived as its multiples.

Also, we introduce the term absolute zero (bolded):

$$\mathbf{0} = 0\overline{0}.$$

Let a, b be any real numbers and $a\overline{0}, b\overline{0}$ two corresponding soft zeros as defined above.

Axiom 1 (Distinction). If $a \neq b$ then $a\overline{0} \neq b\overline{0}$.

In Soft Logic, we extend the zero from a point to a line. This creates a distinction between different multiples of $\overline{0}$.

Definition 1 (Order). If $a < b$ then $a\overline{0} < b\overline{0}$.

Axiom 2 (Addition). $a\overline{0} + b\overline{0} = (a + b)\overline{0}$.

Under the assumption that the multiples of $\overline{0}$ are located on a straight line, we can define the addition of multiples of $\overline{0}$ as the addition of their corresponding real factors. The $\overline{1}$ axis behaves regularly:

$$a\overline{1} + b\overline{1} = (a + b)\overline{1}.$$

Axiom 3 (Nullity). $a\overline{0} * b\overline{0} = 0$.

Numbers on the zero-axis "collapse" under multiplication. Addition has a significance and meaning, but multiplication does not make any distinction whatsoever. On the other hand

$$a\overline{1} * b\overline{1} = ab\overline{1},$$

$$a\overline{0} * b\overline{1} = ab\overline{0},$$

$$b\overline{1} * a\overline{0} = ba\overline{0}.$$

Let us define a value that combines a value from the zero line ($\overline{0}$-axis) with a value from the real line ($\overline{1}$-axis) and denote it $a\overline{0} \perp b\overline{1}$. Note that although the symbol between the two values reminds one of perpendicularity, it nevertheless has quite a different meaning of combining or 'bridge making', and is called by us 'a bridge sign'.

Axiom 4 (Bridging). There exists a bridge between a zero-axis, and a real axis and vice versa, denoted by a pair of a bridge number and its mirror image about the bridge sign:

$$(\boldsymbol{a\overline{0} \perp b\overline{1}, b\overline{1} \perp a\overline{0}}).$$

This is an existence axiom, stating that there is some special connection between two lines in the coordinate system.

Axiom 5 (Non-commutativity).

$$a\overline{0} \perp b\overline{1} \neq b\overline{1} \perp a\overline{0}.$$

Note that the multiplication of a zero-axis number with a regular real number, as defined in Axiom 3, is commutative, but for the subsequent parts of the theory the order of these numbers is important.

The two above combinations, of a soft zero and a real number with a bridge sign between them, will be identified by us as bridge numbers of two different types: a bridge number of a right type (where a real component is to the right of the bridge sign), and a bridge number of a left type (a real component is to the left of the bridge sign). By the Non-commutativity Axiom 5, bridge numbers with the

same components but of different types have different values. When dealing with two bridge numbers of the same type, the operations of addition and multiplication can be defined in the way described in our previous paper.[8] But to define such operations with two bridge numbers of different types is a serious challenge. For example, how to define the sum

$$(a\overline{0}\perp b\overline{1}) + (c\overline{1}\perp d\overline{0}) =?$$

We cannot apply any operations to the right and left appearance of bridge numbers; this is why we need to define the concept of Soft Numbers.

Definition 2 (Soft Number). A Soft Number is a construction of the following form:

$$a\overline{0}\dotplus b\overline{1} = \{(a\overline{0})\perp(b\overline{1}); (b\overline{1})\perp(a\overline{0})\},$$

where a, b are any real numbers.

Any Soft Number in its denomination $a\overline{0}\dotplus b\overline{1}$ has a component from the real line, or $\overline{1}$ line, and a component from the $\overline{0}$ line (later on, we will omit the line above 1, or even 1 itself, but the line over 0 is essential). Note the new operation $\dotplus$ that we define here. The dot above the sign '+' implies that a Soft Number has not one value, as a regular sum, but two distinct values, presented in its definition.

Every Soft Number has a right appearance $a\overline{0}\dotplus b$ and a left appearance $b\dotplus a\overline{0}$. Is the content of a Soft Number influenced by the change of the order of the components in its appearance? This matter is discussed in what follows.

We present a Soft Number in the following way:

$$a\overline{0}\dotplus b = \{c, c'\},$$

where

$$c = a\overline{0}\perp b,$$

$$c' = b\perp a\overline{0}.$$

According to the non-commutativity axiom,

$$a\overline{0}\perp b \neq b\perp a\overline{0},$$

i.e. $c \neq c'$. But as any set is by definition unordered, the pair $\{c, c'\}$ is unordered, i.e.

$$\{c, c'\} = \{c', c\}.$$

Therefore,

$$a\overline{0}\dotplus b = b\dotplus a\overline{0}.$$

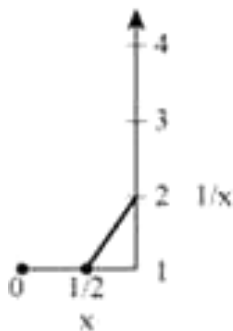By this equation, a Soft Number in both its appearances, the right one and the left one, has the same two values.

Fig. 1. The soft coordinate axis.

In our theory, **SN** denotes the set of all Soft Numbers:

$$\mathbf{SN} = \left\{ a\overline{0} \dotplus b : a, b \in \mathbb{R} \right\}.$$

It is important to remember that any Soft Number in this set has two values, as each number consists of a bridge number $a\overline{0}\perp b$ and its mirror image about the bridge sign $b\perp a\overline{0}$, which have different meanings by the non-commutativity axiom.

3. The Coordinate System of Soft Logic

We suggested in our previous paper[8,10] a new coordinate system, as presented in Fig. 1. It starts from 0 to 1 horizontally and then it turns 90° from 1 to infinity.

Remark 1. There exists a one-to-one correspondence between the segment $(0, 1]$ and the segment $[1, \infty)$.

Remark 2. All lines that connect x to $1/x$ (for all non-zero real x) intersect at a single point. (The statements in Remarks 1 and 2 were demonstrated in our previous paper.[8])

This "single point" denotes the beginning of the soft logic coordinate system. We call this point "the absolute zero". The distance from absolute zero to $+0$ is 1. We suggest extending this new coordinate system to the negative numbers (Fig. 2).

In Fig. 2, we have, in addition to the absolute zero **0,** two additional zeroes. One zero is opposite the number -1, and is not identical with the zero opposite to the number $+1$. Hence, we suggest denoting these two different "zeroes" as $-\overline{\mathbf{0}}$ and $+\overline{\mathbf{0}}$.

The following drawing shows the extended coordinate system for positive and negative numbers with an additional line presenting the multiples of $\overline{0}$ (Fig. 3). The added line is called a zero line or a zero-axis, and the multiples on it are called soft zeros or zero-axis numbers.

The coordinate system in Fig. 3 allows us to present all the real numbers and all the soft zeros. We now wish to construct a coordinate system for representing various Soft Numbers, which may be described as an infinite strip as shown in Fig. 4.
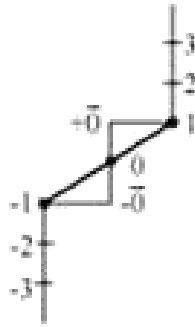
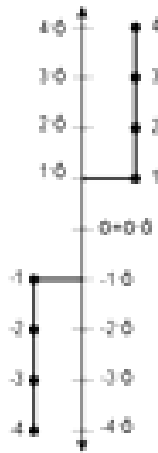Fig. 2. Distinction between -0 and $+0$.

Fig. 3. The extended soft coordinate system.

Because of the Soft Number duality, we double the strip (Fig. 4). This allows us to represent both elements of a Soft Number:

$$c = X\overline{0}\perp Y,$$

$$c' = Y\perp X\overline{0},$$

where X, Y are any real numbers. Each of the elements c and c' is a mirror image of the other about the bridge sign.

Note that we have expanded the coordinate system in Fig. 3 to the one shown in Fig. 4.

As the infinite strip, presented (partially) in Fig. 4, is intended for the presentation of Soft Numbers, we call it a 'Soft Numbers Strip' or briefly, SNS.
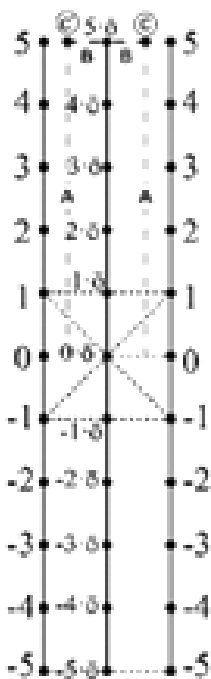
Fig. 4. The complete soft coordinate system.

Definitions. Let C be any point on the SNS:

(1) The **height of the point** C is the vertical distance from C to the horizontal segment with the absolute zero at its center. This distance is supplied with a plus sign if C is above this segment and with a minus sign if C is below it. The height with a sign is denoted by A.

(2) The **width of the point** C is the horizontal distance from C to the zero line and is denoted by B.

The definitions above provide every point C on the SNS with two parameters, A and B (Fig. 4), where

$$-\infty < A < \infty, \quad 0 \leq B \leq 1.$$

The condition $A > 0$ is satisfied in the positive part of the SNS, and $A < 0$ — in its negative part, or correspondingly, above and below the horizontal segment containing the absolute zero, while on this segment $A = 0$.

For the second parameter B there is: $B = 0$ on the zero-axis, $B = 1$ on the lines bounding the SNS, and otherwise $0 < B < 1$.

It is clear that points that are symmetrical about the zero line have the same values of A, B. Thus, a parameter pair (A, B) defines a pair of points on the SNS that are symmetrical about the zero line. The only exception to this rule is the case of $B=0$, in which two points coincide on the zero line.

   The zero line divides the SNS into two symmetric parts: a right one and a left one. Now we want to bring any inner point on the right part of the SNS in correspondence with some bridge number of right type. We exclude from consideration the points on the zero line and on the real line, as they are already filled with numbers. Let C be any point between the zero line and the right real line of the SNS. Then

$$-\infty < A < \infty, \quad 0 < B < 1.$$

A point C with such parameters A, B, is located within a horizontal segment connecting a point $A\overline{0}$ on the zero line with a point $A\overline{1}$ on the right real line, as shown in Fig. 4. Now let us imagine the following: let V be a fixed point on the SNS, different from the three points mentioned above. We consider three vectors $\overrightarrow{r_c}, \overrightarrow{r_0}, \overrightarrow{r_1}$ starting at the point V and ending at the points C, $A\overline{0}$, $A\overline{1}$, respectively. It can be easily verified that vector $\overrightarrow{r_c}$ is a 'convex combination' of vectors $\overrightarrow{r_0}, \overrightarrow{r_1}$, namely,

$$\overrightarrow{r_c} = (1 - B)\overrightarrow{r_0} + B\overrightarrow{r_1},$$

where B is the width of the point C. On the basis of this identity, the following rule is suggested.

   If a point C is on the right side of the zero line and has a height A and a width $B(0 < B < 1)$, it represents the bridge number

$$c = (1 - B)A\overline{0}\perp BA\overline{1},$$

and a point C' on the left side of the zero line, symmetrical to C about this line, represents the mirror image of c about the bridge sign:

$$c' = BA\overline{1}\perp(1 - B)A\overline{0}.$$

The pair of points C, C' represents a pair $\{c, c'\}$ that is a Soft Number

$$(1 - B)A\overline{0}\dotplus BA\overline{1}.$$

To further clarify this, let us see the following example (Fig. 5).

   For each one of the points C, C' in Fig. 5 there is: $A = 3$, $B = 0.7$.

   Hence, the corresponding c, c' are:

$$c = (0.3 * 3)\overline{0}\perp(0.7 * 3)\overline{1} = 0.9\overline{0}\perp2.1,$$

$$c' = (0.7 * 3)\overline{1}\perp(0.3 * 3)\overline{0} = 2.1\perp0.9\overline{0}.$$

Consequently, the Soft Number represented by a given pair of points C, C' is

$$0.9\overline{0}\dotplus2.1.$$

   Thus, we have learned to find a Soft Number represented by a pair of points within the SNS that are symmetrical about the zero line. Let us now review some of the special properties of this number. The real factors of its left and right components are correspondingly $(1 - B)A$ and BA. As the numbers B and $1 - B$ are positive, both real factors have the same sign as the parameter A, or they are both 0 if $A = 0$. These facts involve the following conclusion: Any pair of points on the
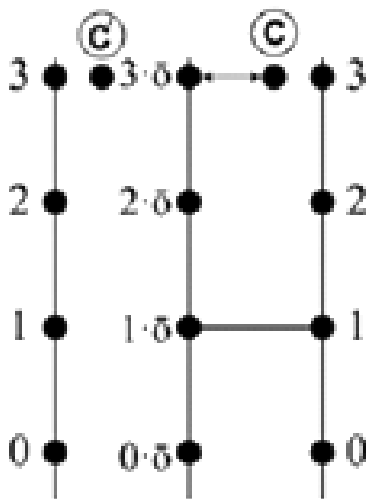
Fig. 5. Presentation of a Soft Number in SNS.

SNS that are symmetrical about the zero line represent a Soft Number $a\bar{0}\dotplus b$ in which the real numbers a, b are of the same sign or are both 0. If a pair of points is located in the positive part of the SNS, then a, b are both positive, and if it is in the negative part, a, b are both negative. If a pair of points lies on a horizontal segment with $A = 0$, then $a = b = 0$, and the Soft Number may be identified as the absolute zero.

In the deliberations above we went from the points to the numbers. Now we will develop an opposite direction.

Given a bridge number c, how can we find the point representing it on the SNS? In other words, given the information regarding the element c of a Soft Number, we wish to calculate A, B. We have

$$c = (1 - B)A\bar{0} \perp BA\bar{1} = X\bar{0} \perp Y\bar{1}.$$

Hence

$$AB = Y,$$

$$(1 - B)A = X.$$

This gives

$$A = X + Y,$$

$$B = \frac{Y}{X + Y}.$$

The calculation for the mirror element

$$c' = Y\bar{1} \perp X\bar{0}$$

leads to the same equations and therefore to the same results for A, B.

Note that the condition $0 < B < 1$ is satisfied if $XY > 0$, i.e. if the real factors on both sides of a Soft Number have the same sign.

From our considerations above we conclude the following: There is a one-to-one correspondence between the set of pairs of points located inside the positive and the negative parts of the SNS that are symmetrical about the zero line, and the set of all Soft Numbers $a\overline{0}\dot{+}b$ with $ab > 0$. Thus, the presentation of Soft Numbers by points on the SNS is found not for the entire set SN, but for the subset of all Soft Numbers that have on both sides real factors of the same sign. Soft Numbers without such a property, e.g. $2\overline{0}\dot{+}(-3)$, cannot be represented on the SNS with the method suggested in this paper. The problem of representing all Soft Numbers on the SNS is still open and may be a theme for further investigation. Here it is to be noted that the method developed in this section (and used below) for assigning Soft Numbers to points on the SNS has a number of advantages, despite its limitations. One of them is the possibility of an immediate and clear visualization, as described in the following remark.

Remark 3. A Soft Number $a\overline{0}\dot{+}b\overline{1} = A(1 - B)\overline{0}\dot{+}AB\overline{1}$ corresponding to a pair of points $\{C, C'\}$ with a height A and a width B on the SNS, may be seen as $S_1\overline{0}\dot{+}S_0\overline{1}$, where S_0, S_1 are the areas of some rectangles originating at point C, as shown in Fig. 8.

Subsequently, we often shorten the term 'representation' to 'presentation' and call the creation of the visual images of Soft Numbers/soft functions on the SNS 'the presentation of Soft Numbers/soft functions on the SNS'.

4. Algebra of Soft Logic

In this section, we develop the algebra of Soft Logic. We will show that it is possible to define a group **SN**$(+)$ and a ring **SN**$(+, \times)$ with a set of Soft Numbers, and that this ring is almost a field. Furthermore, the set of elements of **SN**$(+, \times)$ that are invertible under multiplication is a group that we will denote by **SN**$\ast(\neg 0)$.

4.1. *The operations of addition and multiplication in the set of Soft Numbers*

We define the addition of two Soft Numbers with the following rule:

$$(a\overline{0}\dot{+}b) + (c\overline{0}\dot{+}d) = (a + c)\overline{0}\dot{+}(b + d).$$

With this definition, the set **SN**$(+)$ is a group under addition, because

(1) **SN** is closed under addition.
(2) **SN** has an identity (a neutral) element under addition which is $0 \times \overline{0}\dot{+}0$ (note that this number presents a segment on the **SNS** that is orthogonal to the zero-axis with the absolute zero at its center, if points turn into numbers by the method suggested in the previous section).

(3) Every element in **SN** has an inverse (an opposite) element under addition:

$$(a\overline{0}\dotplus b) + (-a\overline{0}\dotplus(-b)) = 0\overline{0}\dotplus 0.$$

(4) The associativity law is satisfied:

$$((a\overline{0}\dotplus b) + (c\overline{0}\dotplus d)) + (e\overline{0}\dotplus f) = (a\overline{0}\dotplus b) + ((c\overline{0}\dotplus d) + (e\overline{0}\dotplus f)).$$

Proof of this statement is based on the addition definition and can be easily reproduced by the reader, so we do not display it here.

By the exposition above, **SN**$(+)$ is a group. Moreover, it is an Abelian, i.e. a commutative group as the addition definition involves:

$$(a\overline{0}\dotplus b) + (c\overline{0}\dotplus d) = (c\overline{0}\dotplus d) + (a\overline{0}\dotplus b).$$

Now we define multiplication of Soft Numbers by the following rule:

$$(a\overline{0}\dotplus b) \times (c\overline{0}\dotplus d) = (ad + bc)\overline{0}\dotplus(bd).$$

This operation is commutative and satisfies the laws of associativity and distribution. The formulation and checking of the first law is quite clear, and the second law is formulated and checked below. The law of distribution of multiplication relative addition for Soft Numbers:

$$(a\overline{0}\dotplus b) \times ((c\overline{0}\dotplus d) + (e\overline{0}\dotplus f)) = (a\overline{0}\dotplus b) \times (c\overline{0}\dotplus d) + (a\overline{0}\dotplus b) \times (e\overline{0}\dotplus f).$$

The proof of this statement is based on the definitions of addition and multiplication of Soft Numbers as follows:

$$(a\overline{0}\dotplus b) \times ((c\overline{0}\dotplus d) + (e\overline{0}\dotplus f))$$

$$= (a\overline{0}\dotplus b) \times ((c + e)\overline{0}\dotplus(d + f))$$

$$= (ad + af + bc + be)\overline{0}\dotplus(bd + bf)$$

$$= (a\overline{0}\dotplus b) \times (c\overline{0}\dotplus d) + (a\overline{0}\dotplus b) \times (e\overline{0}\dotplus f).$$

With these two operations $+$ and $\times$, Soft Numbers create the Ring **SN**$(+, \times)$.

The Soft Number $0\overline{0}\dotplus 1$ is a neutral element under multiplication in this set because for any of its elements $a\overline{0} \dotplus b$, by the multiplication rule there is:

$$(a\overline{0}\dotplus b) \times (0\overline{0}\dotplus 1) = a\overline{0} \dotplus b.$$

Lemma 1. *If $b \neq 0$, then a Soft Number $a\overline{0} \dotplus b$ has an inverse $\left(-\frac{a}{b^2}\overline{0}\dotplus\frac{1}{b}\right)$.*

To prove this statement, we have to ascertain that the product of both Soft Numbers equals the neutral element of multiplication. By the rule of multiplication, we have

$$(a\overline{0}\dotplus b) \times \left(-\frac{a}{b^2}\overline{0}\dotplus\frac{1}{b}\right) = \left(\frac{a}{b} - \frac{a}{b}\right)\overline{0}\dotplus\frac{b}{b} = 0\overline{0}\dotplus 1.$$

The product is really the neutral element of multiplication, and Lemma 1 statement is proved.

When a common notation for an inverse element is used, it may be written in the following form:

$$(a\overline{0}\dot{+}b)^{-1} = \left(-\frac{a}{b^2}\overline{0}\dot{+}\frac{1}{b}\right).$$

Let us observe the set of all Soft Numbers $a\overline{0}\dot{+}b$ where $b \neq 0$.

The above development demonstrates that this set forms a group under the operation of multiplication, that we denote SN*(¬0).

Therefore, **SN**$(+, \times)$ is almost a field ('almost' — because the inverse is undefined only when $b = 0$).

4.1.1. *Two interesting examples of subgroups of SN* (¬0)*

(1) The set $\mathbf{S}\{N\}^*(\mathbf{1}) = \{a\overline{0}\dot{+}(\pm 1) : a \text{ is any real number}\}$.

The set is closed under multiplication because

$$(a\overline{0}\dot{+}(\pm 1)) \times (c\overline{0}\dot{+}(\pm 1)) = (\pm a \pm c)\overline{0}\dot{+}(\pm 1).$$

(Note that the number 1 in the two factors may have opposite signs)

Every element in this set has an inverse:

$$(a\overline{0}\dot{+}(\pm 1))^{-1} = -a\overline{0}\dot{+}(\pm 1),$$

where the sign of 1 (+ or −) on both sides of '=' is meant to be the same.

(2) The set $\mathbf{S}Q^*(\neg\mathbf{0}) = \{q\overline{0}\dot{+}r \ : q, r \text{ are rational numbers}, r \neq 0\}$.

Since the multiplication and division of rational numbers (with a non-zero denominator) gives a rational number, this set is a subgroup of **SN*** (¬0).

4.2. *The powers and roots of Soft Numbers*

Lemma 2. *The nth power of a Soft Number is*

$$(a\overline{0}\dot{+}b)^n = nab^{n-1}\overline{0}\dot{+}b^n,$$

where n is any natural number $(n = 1, 2, 3, \ldots)$.

We can prove that by induction.

Case $n = 1$ is obvious.

If we assume that the equation in the lemma's formulation is correct for a given natural n, then we can calculate for $n + 1$:

$$(a\overline{0}\dot{+}b)^{n+1} = (a\overline{0}\dot{+}b) \times (nab^{n-1}\overline{0}\dot{+}b^n)$$
$$= (nab^n + ab^n)\overline{0}\dot{+}b^{n+1} = (n+1)ab^n\overline{0}\dot{+}b^{n+1}.$$

We obtain that the equation is valid for $n + 1$, and by the induction principle it is valid for any natural n.

Now we wish to calculate the square root of a Soft Number in the following form:

$$\sqrt{a\bar{0}\dotplus b} = x\bar{0}\dotplus y.$$

Lemma 3. *For $b > 0$ there is*

$$\sqrt{a\bar{0}\dotplus b} = \pm\frac{a}{2\sqrt{b}}\bar{0}\dotplus(\pm)\sqrt{b},$$

where $\sqrt{b}$ denotes an arithmetic (positive) square root of b, and the signs in both components of the soft root above are simultaneously $+$ or simultaneously $-$.

Proof. By using the equation for the nth power above in the case $n = 2$, as a start, and identifying its right part as $a\bar{0}\dotplus b$, we have

$$(x\bar{0}\dotplus y)^2 = 2xy\bar{0}\dotplus y^2,$$

$$2xy = a,$$

$$y^2 = b,$$

$$y = \pm\sqrt{b},$$

$$x = \pm\frac{a}{2\sqrt{b}}$$

and this leads to the square root formula stated in Lemma 3.   □

Obviously, for a Soft Number $a\bar{0}\dotplus b$ with $b < 0$, a square root does not exist.

For $b = 0$, $a \neq 0$ there is no square root as well (it could only be a zero-axis number, and it is impossible to multiply a zero-axis number by itself and to receive a zero-axis number $a\bar{0}$ with $a \neq 0$).

If $a = b = 0$, i.e. a Soft Number is an absolute zero, then its square root is also an absolute zero.

Therefore, there are at most two square roots for every Soft Number.

Lemma 4. *The nth root of a Soft Number satisfies the following statements*:

(1) $\sqrt[n]{a\bar{0}\dotplus b} = \frac{a}{nb^{\frac{n-1}{n}}}\bar{0}\dotplus \sqrt[n]{b}$, *for $b \neq 0$ and an odd n,*

(2) $\sqrt[n]{a\bar{0}\dotplus b} = \pm\frac{a}{nb^{\frac{n-1}{n}}}\bar{0}\dotplus(\pm\sqrt[n]{b})$ *for $b > 0$ and an even n.*

(3) $\sqrt[n]{a\bar{0}\dotplus b}$ *does not exist for $b < 0$ and an even n, and for $b=0$ and $a \neq 0$,*

(4)   *If $a = b = 0$, the root exists and equals the absolute zero.*

Proof. If $\sqrt[n]{a\bar{0}\dotplus b} = x\bar{0}\dotplus y$ then

$$a\bar{0}\dotplus b = (x\bar{0}\dotplus y)^n = nxy^{n-1}\bar{0}\dotplus y^n,$$

which involves

$$y^n = b,$$

$$nxy^{n-1} = a.$$

From the system of these two equations all the statements of Lemma 4 are obtained.

$\square$

5. Soft Logic and Calculus

In this section, we show how Soft Numbers can be used for describing and investigating the tools of calculus, and first of all the most important of them: the derivative.

In the previous section, we received the following result:

$$(a\overline{0}\dotplus b)^n = \mathrm{nab}^{n-1}\overline{0}\dotplus b^n,$$

where n is any natural number. With the use of the formula for a derivative of the power function of a real variable

$$(x^n)' = nx^{n-1},$$

the power function of a soft variable $1\overline{0}\dotplus x$ with a real component x can be written as

$$(1\overline{0}\dotplus x)^n = nx^{n-1}\overline{0}\dotplus x^n = (x^n)'\overline{0}\dotplus x^n.$$

Generally, the power function of any soft variable is:

$$(a\overline{0}\dotplus x)^n = nax^{n-1}\overline{0}\dotplus x^n = (ax^n)'\overline{0}\dotplus x^n.$$

Let us observe a polynomial function of a real variable:

$$P(x) = \sum_{k=0}^{n} a_k x^k.$$

A polynomial function of a soft variable $1\overline{0}\dotplus x$ is

$$P(1\overline{0}\dotplus x) = \sum_{k=0}^{n} a_k(1\overline{0}\dotplus x)^k = \sum_{k=0}^{n} ka_k x^{k-1}\overline{0} + \sum_{k=0}^{n} a_k x^k.$$

By comparing the two polynomial functions above, we come to the fundamental formula connecting Soft Numbers with calculus, as presented in what follows.

Lemma 5. *If $P(x)$ is a real polynomial function, then*

$$P(1\overline{0}\dotplus x) = P'(x)\overline{0}\dotplus P(x)$$

and, generally, for any Soft Number $a\overline{0}\dotplus x$, there is

$$P(a\overline{0}\dotplus x) = aP'(x)\overline{0}\dotplus P(x).$$

If $f(x)$ is an analytic function, it can be approximated by its Taylor polynomials $P_n(x)$ so that we have

$$\lim_{n\to\infty} P_n(x) = f(x), \ \lim_{n\to\infty} P_n'(x) = f'(x).$$

On the ground of these limit conditions and the statements of Lemma 5, we arrive at the following definition of a function of a soft variable that is arising from the real differentiable function $f(x)$ and that we denote $\overline{f}$:

$$\overline{f}(1\overline{0}\dotplus x) := f'(x)\overline{0}\dotplus f(x)$$

or, generally, for any Soft Number $a\overline{0}\dotplus x$:

$$\overline{f}(a\overline{0}\dotplus x) := af'(x)\overline{0}\dotplus f(x).$$

This formula may be seen as an analog of Lagrange's formula in the world of Soft Numbers. It extends a real differentiable function f to a soft function $\overline{f}$.

We can use Soft Numbers to develop the product rule and the chain rule of derivatives:

$$(f \times g)(1\overline{0} \dotplus x) = f(1\overline{0} \dotplus x) \times g(1\overline{0} \dotplus x)$$

$$= (f'(x)\overline{0} \dotplus f(x)) \times (g'(x)\overline{0} \dotplus g(x))$$

$$= (f'(x) \times g(x) + f(x) \times g'(x))\overline{0} \dotplus f(x) \times g(x).$$

Hence we obtain the product derivative rule:

$$(f \times g)' = f' \times g + f \times g'.$$

The chain rule for the derivative of a composed function is received as follows:

$$(f \circ g)(1\overline{0} \dotplus x) = f(g(1\overline{0} \dotplus x))$$

$$= f(g'(x)\overline{0} \dotplus g(x)) = f'(g(x))g'(x)\overline{0} \dotplus f(g(x)).$$

Hence

$$(f \circ g)' = g' \times f'(g).$$

6. Soft Functions

In this section, we use two different ways of extending a real function $f : R \rightarrow R$ to a soft function. The first way of extension provides the function $\overline{f} : \mathrm{SN} \rightarrow \mathrm{SN}$ of a soft variable with soft values. In the second way, we obtain the extended function $\hat{f} : R \rightarrow \mathrm{SN}$ of a real variable with soft values.

The first way of extension can be applied only to such functions $f(x)$ that are differentiable. Given a real differentiable function f, we will define its soft extension $\overline{f}$ with the following formula (x, y are any real numbers in the domain of f):

$$\overline{f}(x\overline{0}\dotplus y) := xf'(y)\overline{0}\dotplus f(y).$$

6.1. *Fundamental properties of soft functions*

In this subsection, we consider the functions $\overline{f} : \mathrm{SN} \rightarrow \mathrm{SN}$.

Lemma 6. *For any two differentiable real functions f, g there is*

(a)
$$\overline{f + g} = \overline{f} + \overline{g},$$

(b)
$$\overline{f \bullet g} = \overline{f} \bullet \overline{g},$$

(c)
$$\overline{f \circ g} = \overline{f} \circ \overline{g},$$

(d)
$$\overline{\left[\frac{1}{f}\right]} = \frac{1}{\overline{f}} \ , \ (f \neq 0).$$

Remark. A quotient with the number 1 as a nominator and a Soft Number as a denominator denotes the inverse of this Soft Number.

Proof of Lemma 6.

(a)
$$\overline{f + g}(x\overline{0}\dot{+}y) = x(f + g)'(y)\overline{0}\dot{+}(f + g)(y) = (xf'(y) + xg'(y))\overline{0}\dot{+}(f(y) + g(y))$$
$$= (xf'(y)\overline{0}\dot{+}f(y)) + (xg'(y)\overline{0}\dot{+}g(y)) = \overline{f}(x\overline{0}\dot{+}y) + \overline{g}(x\overline{0}\dot{+}y).$$

Therefore, $\overline{f + g} = \overline{f} + \overline{g}$.

(b)
$$\overline{f \cdot g}(x\overline{0}\dot{+}y) = x(fg)'(y)\overline{0}\dot{+}(fg)(y) = (xf'(y)g(y) + xf(y)g'(y))\overline{0}\dot{+}f(y)g(y)$$
$$= (xf'(y)\overline{0}\dot{+}f(y)) \cdot (xg'(y)\overline{0}\dot{+}g(y)) = \overline{f}(x\overline{0}\dot{+}y) \cdot \overline{g}(x\overline{0}\dot{+}y).$$

Therefore, $\overline{f \cdot g} = \overline{f} \cdot \overline{g}$.

(c)
$$(\overline{f} \circ \overline{g})(x\overline{0}\dot{+}y) = \overline{f}(xg'(y)\overline{0}\dot{+}g(y)) = xg'(y)f'(g(y))\overline{0}\dot{+}f(g(y))$$
$$= x(f \circ g)'(y)\overline{0}\dot{+}(f \circ g)(y) = \overline{f \circ g}(x\overline{0}\dot{+}y).$$

Therefore, $\overline{f \circ g} = \overline{f} \circ \overline{g}$.

(d)
$$\overline{\left[\frac{1}{f}\right]}(x\overline{0}\dot{+}y) = x(\frac{1}{f})'(y)\overline{0}\dot{+}(\frac{1}{f})(y)$$
$$= -x\frac{f'}{f^2}(y)\overline{0}\dot{+}\frac{1}{f(y)}$$
$$= \frac{1}{xf'(y)\overline{0}\dot{+}f(y)} = \frac{1}{\overline{f}(x\overline{0}\dot{+}y)}.$$

Thus, it is proved that

$$\overline{\left[\frac{1}{f}\right]} = \frac{1}{\overline{f}},$$

where $f \neq 0$, and this completes Lemma 6 proof.   □

Before formulating the next statement, we wish to remind the concept of an inverse function. Let f, g be real functions, mapping, each one, R on R. Let e be the identity function in R, mapping every real number into itself, i.e. $e(x) = x$ for every real x. The function g is said to be an inverse function for f and denoted $g = f^{-1}$, if and only if $f \circ g = e$.

Let F, G be soft functions, mapping, each one, SN on SN. Let I denote the identity function in SN which maps every Soft Number into itself, i.e. $I(x\overline{0} \dotplus y) = x\overline{0} \dotplus y$ for any real x, y. It is easy to verify that $I = \overline{e}$.

The soft function G is said to be an inverse function for the soft function F and denoted: $G = F^{-1}$, if and only if

$$F \circ G = I.$$

Lemma 7. *If $g = f^{-1}$ in R then $\overline{g} = (\overline{f}\,)^{-1}$ in SN.*

Proof. If $g = f^{-1}$ then $f \circ g = e$ and consequently

$$\overline{f \circ g} = \overline{e} = I.$$

By Lemma 6(c), $\overline{f \circ g} = \overline{f} \circ \overline{g}.$

Combining this equation with a previous one, we obtain

$$\overline{f} \circ \overline{g} = I,$$

which means that $\overline{g} = (\overline{f}\,)^{-1}$.   □

Lemma 8. *For any two Soft Numbers $\alpha = a\overline{0} \dotplus b$, $\beta = c\overline{0} \dotplus d$ there is*

$$\overline{e^{\alpha+\beta}} = \overline{e^{\alpha}}\,\overline{e^{\beta}}.$$

Proof. According to the rule of an extension of a real function to a soft one, we have on one side:

$$\overline{e^{\alpha+\beta}} = \overline{e^{(a\overline{0} \dotplus b)+(c\overline{0} \dotplus d)}} = \overline{e^{(a+c)\overline{0} \dotplus (b+d)}} = (a+c)e^{b+d}\overline{0} \dotplus e^{b+d}$$

and on the other side:

$$\overline{e^{\alpha}}\,\overline{e^{\beta}} = \overline{e^{(a\overline{0} \dotplus b)}}\,\overline{e^{(c\overline{0} \dotplus d)}} = (ae^{b}\overline{0} \dotplus e^{b})(ce^{d}\overline{0} \dotplus e^{d})$$

$$= (a+c)e^{b+d}\overline{0} \dotplus e^{b+d}.$$

By comparing the two results above, we conclude $\overline{e^{\alpha+\beta}} = \overline{e^{\alpha}}\,\overline{e^{\beta}}$.   □

6.2. *Soft curve*

In this subsection, we use the second way of extension of a real function f to a soft function of a real variable

$\hat{f} : R \rightarrow$ SN, performed by the formula:

$$\widehat{f}(x) = x\overline{0} \dot{+} f(x),$$

which does not require a differentiability of f. Following this way, we construct a graphic presentation of f in the Soft Numbers coordinate system, defined in the **SNS** — the strip shown in Fig. 4.

By the method developed in Sec. 3, any Soft Number $x\overline{0}\dot{+}y$ with $xy > 0$ is represented on the **SNS** as a pair of points that are symmetrical about the zero-axis and having the same values of a height A and a width B:

$$A = x + y,$$

$$B = \frac{y}{x + y}.$$

Applying this assertion to a soft function $\widehat{f}(x)$, we come to the following conclusion: any soft function $\widehat{f}(x) = x\overline{0} + f(x)$ of a real variable x, satisfying the condition

$$xf(x) > 0$$

can be visually presented on the **SNS** as two curves that are symmetrical about the zero-axis and are described, each one, by the following equations:

$$A = x + y = x + f(x),$$

$$B = \frac{y}{x + y} = \frac{f(x)}{x + f(x)}.$$

(Note that $y=f(x)$ is the equation of an original real function $f(x)$ generating a soft function $\widehat{f}(x)$).

As $\widehat{f}(x)$ is a soft extension of $f(x)$, the presentation of a soft function $\widehat{f}(x)$ on the SNS will be called by us a presentation of a real function $y = f(x)$ on the **SNS** or briefly, a soft presentation of $y = f(x)$. For the construction of such a presentation we define an orthogonal (A, B) coordinate system on the SNS, where A is a height, B is a width of a point on the SNS, the A-axis coincides with the zero line and the B-axis is a horizontal segment starting at the absolute zero and continuing to the right real line.

Let us expose two examples of constructing a sift graphic presentation of real functions in the (A, B) coordinate system on the SNS.

Example 1. $f(x) = kx$, where $k>0$ and x satisfies the condition:

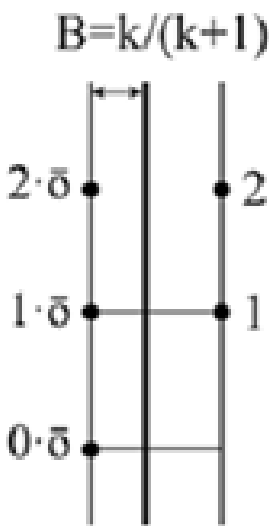$$xf(x) > 0 \Leftrightarrow kx^2 > 0 \Leftrightarrow x \neq 0.$$

Fig. 6. The soft curve $y = f(x) = kx$.

For any $x \neq 0$ we have

$$A = x + kx = (k+1)x,$$

$$B = \frac{kx}{x + kx} = \frac{k}{k+1}.$$

Because B is constant, we obtain two straight lines on the **SNS** that are parallel to the zero-axis at the distance $B = \frac{k}{k+1}$ from it.

In Fig. 6, we can see the right part of the presentation of $f(x) = kx$, $k > 0$, in the soft coordinate system.

Example 2. $f(x) = kx + d$, where $k > 0$, $d > 0$,

$$x > 0 \quad \text{or} \quad x < -\frac{d}{k}.$$

Under these conditions, we have

$$A = (k+1)x + d,$$

$$B = \frac{kx + d}{(k+1)x + d}.$$

From these two equations, A can be expressed in the terms of B by the following calculation:

$$x = \frac{A - d}{k + 1} \Rightarrow AB = \frac{k(A - d)}{k + 1} + d \Rightarrow A = \frac{d}{(k+1)B - k}.$$

The final equation leads to the conclusion that the right part of the graphic presentation of the function $f(x) = kx + d$ on the **SNS** under the given restrictions is a hyperbolic curve with two branches and a vertical asymptote $B = \frac{k}{k+1}$.
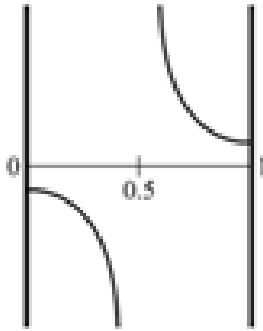
Fig. 7. The soft curve $y = kx + d$.

In the case $k = d = 1$ the result looks like this (Fig. 7).

Examples 1 and 2 show that the development of the equation of the soft graphic presentation of $f(x)$ takes time and effort, even in the case of simple functions $f(x)$. This problem is reduced by the following Lemma 9.

Lemma 9. *Let $f(x)$ be any real function satisfying the condition $xf(x) > 0$ in some real domain D. Then in this domain, the function $f(x)$ can be presented on the SNS with two curves that are symmetric about the zero-axis and that have in the coordinates A, B, the following equation*:

$$AB = f(A - AB).$$

If the function f has an inverse function f^{-1}, then the equation above is equivalent to the equation:

$$A - AB = f^{-1}(AB).$$

Proof. By definition,

$$A = x + f(x),$$

$$AB = f(x).$$

These equations involve:

$$x = A - AB,$$

$$f(x) = f(A - AB).$$

Substituting this expression for $f(x)$ into the equation $AB = f(x)$, we receive

$$AB = f(A - AB).$$

Thus, we obtain the first equation of the lemma. If the function f has an inverse function, then by applying it to both sides of the above equation we receive the second equation of the lemma, which is equivalent to the first one. $\square$

As the first equation in Lemma 9 is valid for any real function *f*, we will call it 'a general (A, B)-equation of $f(x)$', while the second equation will be called 'an inverse (A, B)-equation of $f(x)$'.

Let us demonstrate the use of Lemma 9 for the functions in Examples 1 and 2 above.

The general (A, B)-equation of $f(x) = kx \ (k > 0)$ is

$$AB = k(A - AB) \Rightarrow B = \frac{k}{k+1}.$$

The general (A, B)-equation of $f(x) = kx + d(k > 0, \ d > 0)$ is

$$AB = k(A - AB) + d \Rightarrow A = \frac{d}{(k+1)B - k}.$$

We see that with the help of Lemma 9 the final equations in Examples 1 and 2 are obtained faster and easier than before, without it. Therefore, in the next examples we will use Lemma 9 as a permanent tool for obtaining (A, B)-equations of given real functions. But before this, we wish to note several important properties of the equations stated in Lemma 9.

Remark 1. The equations in Lemma 9 are implicit. If A can be expressed in terms of B, then equivalent explicit equations are obtained. Either an implicit or an explicit form of an equation with two variables A, B, allows to construct corresponding curves on the SNS.

Remark 2. The general (A, B)-equation of $f(x)$ can be received from the original equation $y=f(x)$ by the substitutions $x = A - AB, y = AB$. These are the formulas suggested in Sec. 3, by which a Soft Number $x\overline{0}\dotplus y$ is assigned to any pair of points C, C' with a height A and a width B on the SNS outside the axis.

Remark 3. A general (A, B)-equation of a real function f on the SNS can be seen as a very simple equation between some areas in the SNS. Let $C \ (A, B)$ be any fixed point in the (A, B) coordinate system on the right part of the SNS, with $A > 0$ or $A < 0$ and $0 < B < 1$. Two segments containing the point C can be drawn: a horizontal segment going through the point C and connecting the zero line with the right real line, and a vertical segment going from the point C to the segment $A = 0$.

With this drawing, two rectangles are created. The left rectangle has a vertical side with a length A on the zero line, and the length of its horizontal side is B. The right rectangle has a vertical side with a length A on the right real line, and the length of its horizontal side is 1-B. Let S_0, S_1 denote the areas of these rectangles, as shown in Fig. 8.
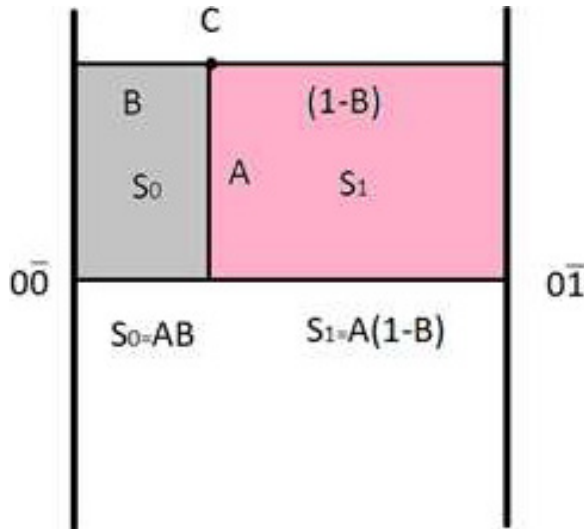
Fig. 8. Rectangles of the soft coordinate system.

The above construction enables visualizing the general (A, B)-equation of a real function f, as stated in the following Lemma.

Lemma 10. *A general (A, B)-equation on the SNS of any real function f can be written in the form*:

$$S_0 = f(S_1),$$

where S_0, S_1 are the areas shown in Fig. 8 (if $A < 0$, the areas are denoted with a minus sign).

Proof. The definition of S_0, S_1 involves that

$$S_0 = AB, S_1 = A(1 - B)$$

and therefore the equation $S_0 = f(S_1)$ is just another (visual) form of the general (A, B)-equation of f, stated in Lemma 9, and thus justifies the statement of Lemma 10 is proved.

It is worth noting here that the areas indicated in Fig. 8 also visualize the real factors of the Soft Number $x\overline{0}\dotplus y$, which is represented by the pair of points on the SNS with a height A and a width B. Namely, $x = S_1$, $y = S_0$ (see Remark 3 in Sec. 3).

Now we can expose and discuss examples of constructing soft presentations of real functions by exploring Lemma 9. At this stage, two prevenient remarks are needed:

(1) Any function $f(x)$ in the examples below is considered under the condition $xf(x) > 0$ that guarantees the existence of the parameters A, B. Under this

condition, the graphic presentation of $f(x)$ on the SNS is located in the inner space between the lines $B = 0$, $B = 1$, $A = 0$. Nevertheless, some curves in the following examples contain points on these lines. Such points can be seen in quite a regular way as limit positions of the inner presentation, when one of the variables $x, y = f(x)$ or both of them are tending to 0.

(2) Any soft graphic presentation of a real function on the SNS contains two curves, one to the right of the zero-axis, and one to the left of it, symmetrical about it. In each one of the figures below only the right part of the presentation is shown.   □

Example 3. $y = x^i$, where $i = 2, 3, 4, \ldots$.

In this case, $f(x) = x^i$ and the admissible values of x are

$$xf(x) > 0 \Leftrightarrow x^{i+1} > 0 \Rightarrow x \neq 0 \quad \text{for an odd } i,$$

$$x > 0 \quad \text{for an even } i.$$

By Lemma 9, the general (AB)-equation for the given function is

$$AB = (A - AB)^i.$$

From this equation we can, by obvious algebraic transformations, obtain the explicit expression:

$$A = {}^{i-1}\!\sqrt{\frac{B}{(1 - B)^i}}.$$

In Fig. 9, we see soft graphs created by this equation on the **SNS** for $i = 2, 3, \ldots, 10$. For $i = 3, 5, 7, 9$, the degree $i-1$ of the root in the formula for A is even, and therefore every one of the four graphs corresponding to these i has two branches: a positive one and a negative one, symmetrical about the segment $A = 0$.

Example 4. $y = \log_i x$, $i = 2, 3, 4, \ldots$.

In this case, $f(x) = \log_i x$, and the condition $xf(x) > 0 \Leftrightarrow x\log_i x > 0$ is satisfied if and only if $x > 1$.

For such values of x, by Lemma 9, the general (A, B)-equation of $f(x)$ is

$$AB = \log_i(A - AB).$$

This implicit equation, for $i = 2, 3, \ldots, 10$, defines nine curves on the **SNS**, as shown in Fig. 10.

Example 5. $y = \tan x$.

In this case, the admissible values of x are

$$xf(x) > 0 \Rightarrow x\tan x > 0 \Leftrightarrow k\pi < x < k\pi + \frac{\pi}{2},$$

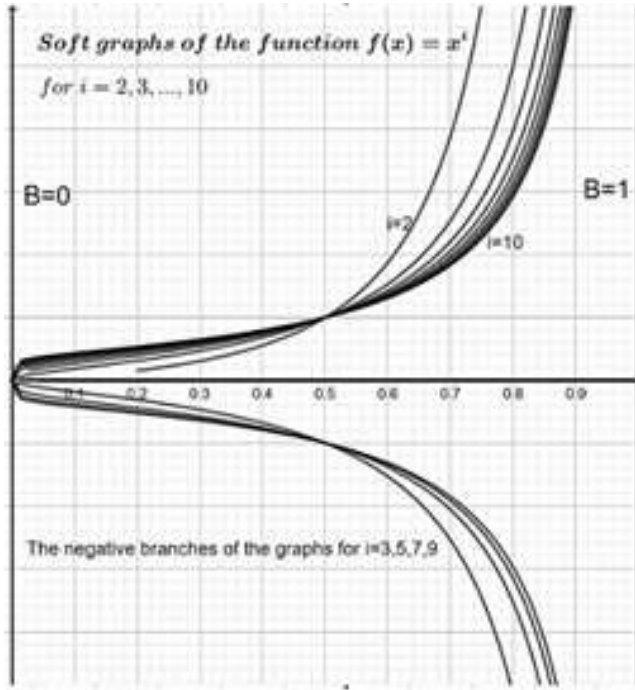$$-k\pi - \frac{\pi}{2} < x < -k\pi \quad (k = 0, 1, 2, 3, \ldots).$$

Fig. 9. The soft curve $y = x^i$.

For any such x the values of a height A and a width B do exist and are defined by the following formulas:

$$A = x + \tan x,$$

$$B = \frac{\tan x}{x + \tan x}.$$

Conceiving x as a parameter and the two equations above as a parametric description of the graphic presentation of $y = \tan x$ in the (A, B) coordinate system on the **SNS**, we can investigate the behavior of this presentation in any of the intervals indicated above. The investigation results are as follows:

(1) In the interval $0 < x < \frac{\pi}{2}$ the graph ascends from the point $A = 0$, $B = 1/2$ to the line $B = 1$ as an asymptote.
(2) In every interval $k\pi < x < k\pi + \frac{\pi}{2} (k = 1, 2, 3, \ldots)$ the graph ascends from the point $A = k\pi$, $B = 0$ to the line $B = 1$ as an asymptote.
(3) In every interval $-k\pi - \frac{\pi}{2} < x < -k\pi$ the graph is symmetric, about the segment $A = 0$, to the graph in the interval $k\pi < x < k\pi + \frac{\pi}{2}$

$$(k = 0, 1, 2, 3, \ldots).$$
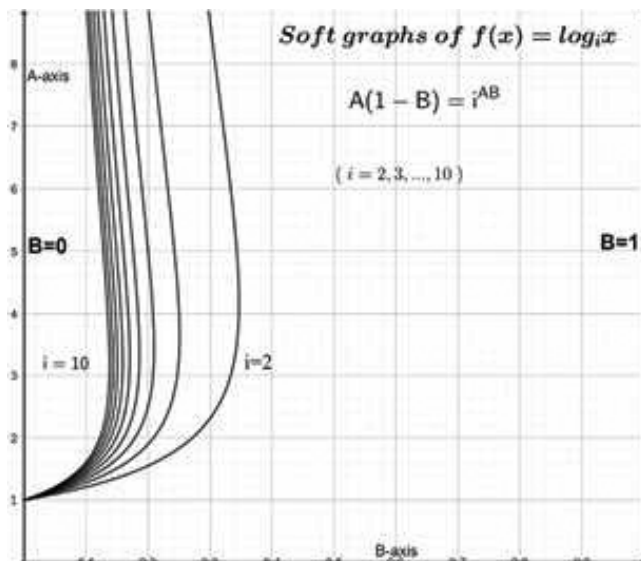
Fig. 10. The soft curve $y = \log_i x$.

(The interested reader is invited to check these assertions.) As the number of intervals of admissible values of x for $\tan x$ is infinite, there are infinitely many curves in the soft graphic presentation of $y = \tan x$ on the **SNS**. In Fig. 11, only a finite part of the whole picture on the **SNS**, to the right of the zero-axis, is shown.

The separate branches of the soft graph $y = \tan x$ are created with the help of the inverse (A, B)-equations:

$$A - AB = \arctan(AB) \ + m\pi \quad (m = 0, \pm 1, \pm 2, \ldots).$$

Example 6. $f(x) = e^x$.

The general (A, B)-equation of this function in the domain $x > 0$ is

$$AB = e^{A - AB}.$$

The soft graph of this function on the SNS (to the right of the zero-axis) is presented in Fig. 16.

Example 7. $f(x) = \sqrt{x}$.

The general (A, B)-equation of this function in the domain $x > 0$ is

$$AB = \sqrt{A - AB}$$

and its inverse (A, B)-equation is

$$A - AB = (AB)^2.$$

The soft graph of $f(x) = \sqrt{x}$ is exposed in Fig. 12.
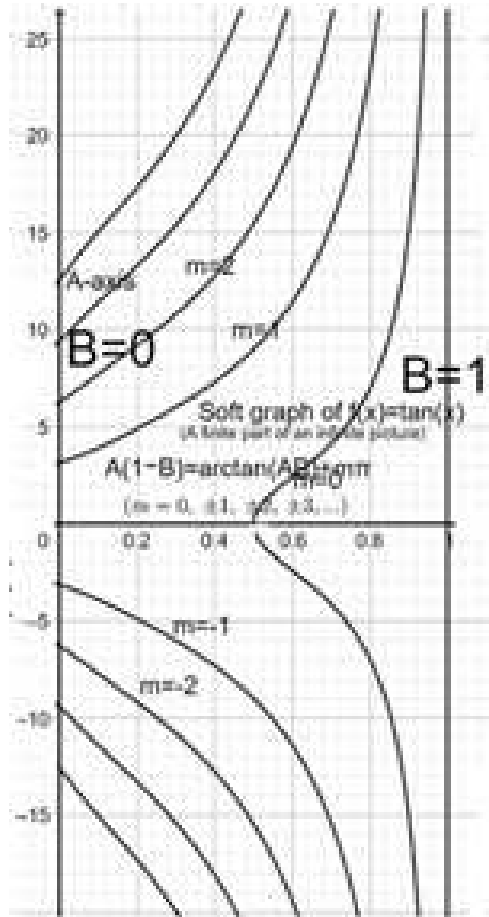
Fig. 11. The soft curve $y = \tan x$.

Example 8. $f(x) = \frac{1}{x}$.

The general (A, B)-equation of this function in the domain $R \backslash \{0\}$ is

$$AB = \frac{1}{A - AB},$$

which is equivalent to

$$A^2 B - A^2 B^2 = 1.$$

The right part of the corresponding soft graphic presentation on the SNS is shown in Fig. 13.

Example 9. $f(x) = \frac{x}{1+x^2}$.

It is an odd continuous function on R, satisfying the condition $xf(x) > 0$ for all real non-zero x. In this domain, it has no inverse function and consequently no
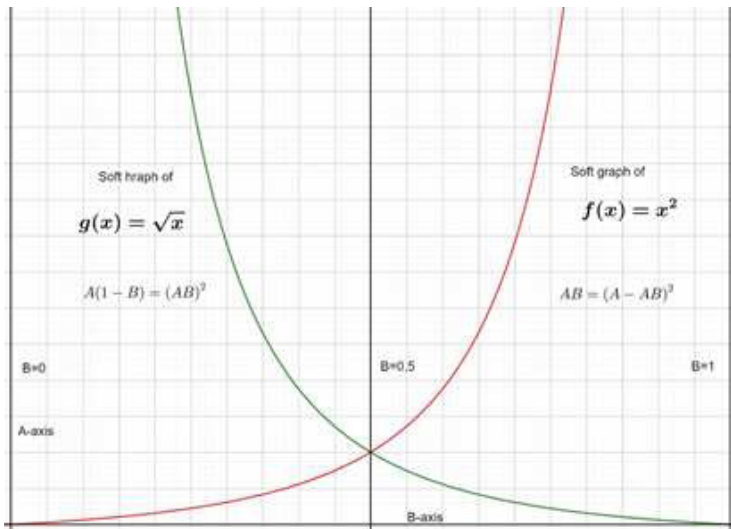
Fig. 12. The soft curve $f(x) = \sqrt{x}$.
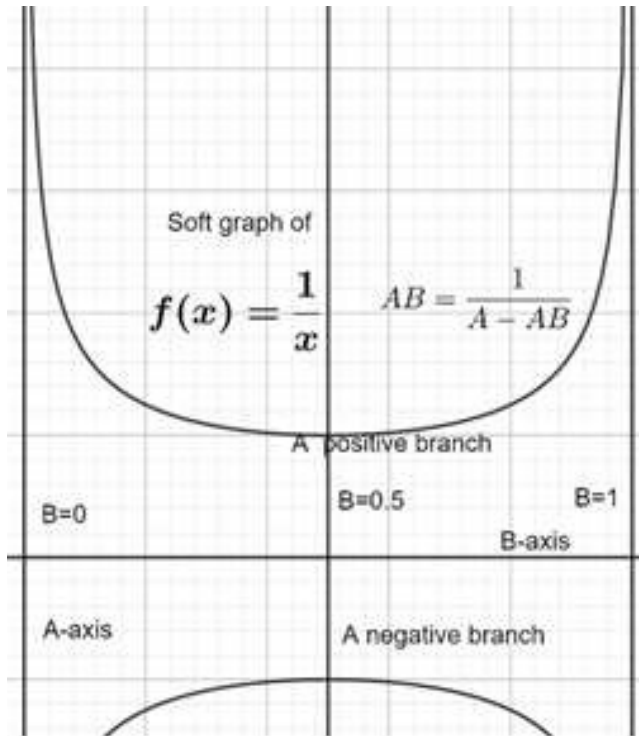
Fig. 13. The soft curve $f(x) = \frac{1}{x}$.
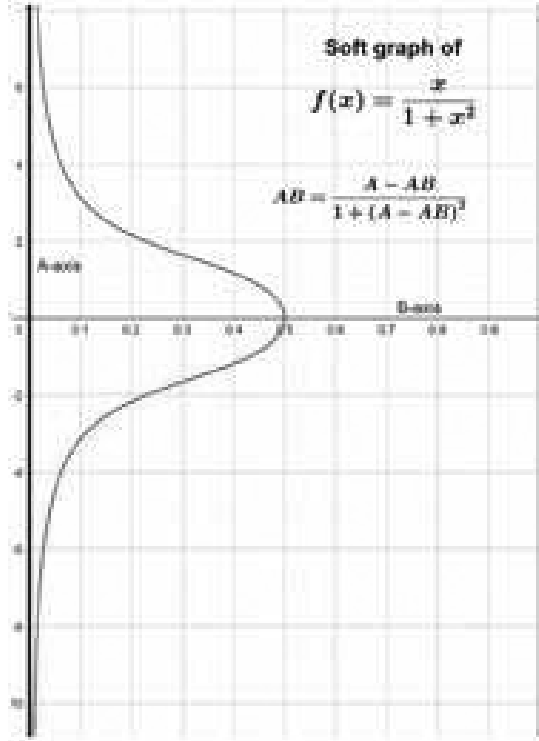
Fig. 14. The soft curve $f(x) = \frac{x}{1+x^2}$.

inverse (A, B)-equation. The general (A, B)-equation of this function is

$$AB = \frac{A - AB}{1 + (A - AB)^2}.$$

The corresponding soft presentation on the SNS, to the right of the zero-axis, is shown in Fig. 14.

6.3. *The symmetry in a soft graphic presentation of two mutually inverse functions*

In this subsection, we will discuss the graphic presentation on the SNS of two real functions that are inverse to each other, e.g. $f(x) = e^x$, $g(x) = \ln x$. In the Cartesian coordinate system (x,y) their graphs are symmetric about the line $y = x$ (Fig. 15).

Figure 16 shows the graphic images of the same two functions in the Soft Numbers coordinate system.

We can see that there is a mirror symmetry about the vertical line $B = 1/2$ in the soft graphic presentations of these two functions. It may be verified that this situation is valid for any two mutually inverse real functions.
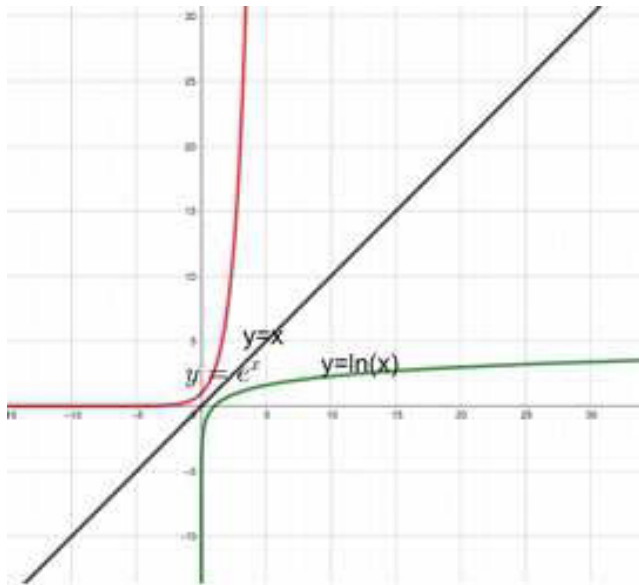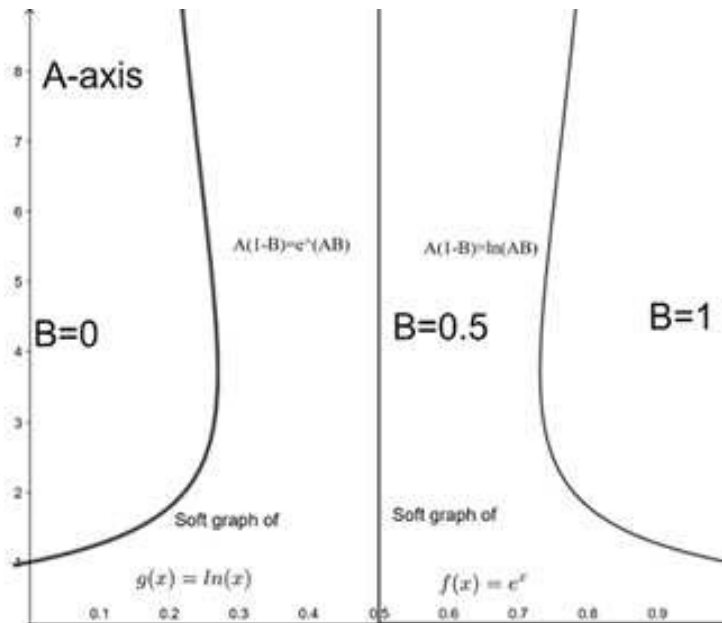
Fig. 15. $f(x) = \ln x,\ g(x) = e^x$.

Fig. 16. $\ln x,\ g(x) = e^x$, in soft coordination.

Lemma 11. *If f, g are two real functions, inverse to each other, then the right/left parts of their soft presentations in the same SNS are symmetric about the vertical line $B = 1/2$.*

Proof. By Lemma 9, two equivalent (A, B)-equations of the soft graph of $f(x)$ on the SNS are

$$AB = f(A(1 - B)),$$

$$A(1 - B) = g(AB)$$

and two equivalent (A, B)-equations of the soft graph of $g(x)$ are

$$AB = g(A(1 - B)),$$

$$A(1 - B) = f(AB).$$

It can be easily seen that the mapping $(A, B) \to (A, 1 - B)$ transforms one pair of the above equations into the other pair. As such a mapping transfers any point in the (A, B)-coordinate system into a point symmetric to it about the line $B = 1/2$, the conclusion is that the curves representing two mutually inverse functions f, g in the right/left part of the SNS are symmetric about the vertical line $B = 1/2$, as stated in Lemma 11. □

To resume and exemplify this statement, let us return to the functions $f(x) = e^x$, $g(x) = \ln x$ that are inverse to each other. The curves representing them on the SNS are symmetric about the line $B = 1/2$ (Fig. 16).

The same situation exists for the functions

$$f(x) = x^2, \quad g(x) = \sqrt{x}.$$

In the domain $x > 0$ (Fig. 12).

The case of the function

$$f(x) = \frac{1}{x}$$

is special, because this function and its inverse function are identical:

$$f^{-1}(x) = f(x) = \frac{1}{x}.$$

Therefore, the soft graphic presentation of

$$f(x) = \frac{1}{x}$$

in the right/left part of the SNS is itself symmetric about the line $B = 1/2$ (Fig. 13).

7. Conclusions and Further Research Topics

In this paper, we complete the Soft Logic system by creating a double-sided bridge between the zero line and the real line, with the help of special tools named 'Soft

Numbers'. This model is relevant to constructing a bridge between the real world and the world of information, which is relevant to our world today. We present the algebra of Soft Numbers and their applications to calculus and to the extension of real functions.

There are some similarities and analogies between Soft Numbers and complex numbers. Imaginary numbers were invented to resolve an unsolved exercise, $i = \sqrt{-1}$, and Soft Numbers were developed to resolve the unsolved problem of $0/0$. In the beginning, when complex numbers were first introduced, there was great resistance to this new concept, as mathematicians could not accept the concept of a new number that is not real. The praxis of algebra with complex numbers certainly helped to make a place for them in the research of mathematics. A very important step in their acceptance was the discovery of a link to the practical field of electricity, when it was shown that one can measure the phase of an electric current with complex numbers. At that point the importance of complex numbers became obvious. There is another analogy between the geometry of Soft Logic and non-Euclidian geometry. In his book The Elements, Euclid established the geometry of the plane by declaring the five axioms. He himself was not satisfied with the 5th axiom, which is called the parallels axiom: "In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point." During the 19th century, several mathematicians developed a non-Euclidian geometry that satisfied the first 4 axioms of geometry of Euclid and does not satisfy the 5th axiom. One of them was Riemann, who developed a non-Euclidian geometry in which there are no parallel lines at all. In 1915, Albert Einstein applied Riemanns new geometry in the theory of general relativity. The development of quantum theory and relativity theory during the 20th century raised the need to develop a more complex or strange geometry. David Bohm developed the concept of non-locality. The question is how to develop a geometry of the universe that enables embedding non-locality so that transmission can be faster than the speed of light? The Lorenz transformation is another analog to the Soft Logic new coordinate system. Lorenz developed his famous transformation in an effort to explain the Michelson Morley experiment to measure the speed of the planet earth relative to the ether. The negative result of the experiment brought Lorenz to develop a family of linear transformations from a coordinate frame in space-time to another frame that moves at a constant velocity relative to the former. Einstein used the Lorenz transformation to develop the theory of special relativity, with the axiom that the speed of light is the fastest velocity in nature. Note that the distance between two different points on the zero line is 0 which enables immediate transportation (non-locality).

We believe that the new Soft Logic coordinate system may be relevant for the description of a non-locality in nature, in which there is a dimension for transportation which is faster than light. Some evidence and support may be obtained from the modern theories of black holes, which use Einsteins equations for general relativity with the expression R/R, where $R = 0$.

Finally, we would like to mention the similarity between Soft Logic and fractal theory which was developed by Mandelbrot and other. This theory break the concept of linearity in nature.

Here is a short list of some of our major topics for investigation in further research:

(1) The connection between Soft Numbers and linear algebra, in particular in matrix theory.

We can define a matrix A and a vector V over the set of Soft Numbers. The question here is: How to solve equations of the type $Ax = V$ in Soft Numbers?

Is the fact that SN is almost a field relevant to our definitions of "Soft Matrices" and "Soft Vectors"?

(2) The geometry of a Soft Number and its connection to the Mobius strip, Klein bottle,[11,12] and Necker cube.

(3) The dynamical system of a soft function when we apply this function to a given Soft Number repeatedly, performing an iterative process. We wish to investigate the tendencies of this process.

(4) The processes within the zero-axis are not measured but affect the real values axis. We will study this phenomenon from the point of views of information theory and modern evolving view of the theory of consciousness.

(5) Developing the concept of highly coupled Cyber-Physical Systems (CPSs). These are complex systems that have two intertwined components: a physical part and a computer part that drives the system. They receive input from the environment (usually sensory saturated) and perform a time-driven task. Examples are: automated vehicles (which we will focus on), fighter planes (autonomous robots), IoT (Internet of Things), etc.

The first building block is to define the state of the system, which comprises the computer state and the physical state (for example the program and the Newtonian physics of the car). From the states, we will mathematically describe the system's evolution, then the control and decision\optimization, followed by networking, reliability (risk), security, and scalability considerations. We are interested in the emerging properties of such systems, for example avoiding congestions by optimal planning, and phase transitions.

(6) Every Soft Number $a\overline{0} \dotplus b$ has two appearances, a right one and a left one. When we look from the other side of the strip of Soft Numbers (the SNS), we can extend the number of appearances of a very Soft Number from 2 to 4, relatively to the four different types of its bridge elements: right-up, left-up, right-down, left-down, as follows:

$$a\overline{0} \perp b,$$
$$b \perp a\overline{0},$$
$$a\overline{0} \top b,$$
$$b \top a\overline{0}.$$

Recursive Distinctioning (RD) is an invention of Isaacson,[13] in which he describes how fundamental patterns arise from the operation of distinction by recursion, with the use of four SA letters {=, [,], O}. We wish to investigate the connection between Soft Numbers and RD in the context of developing a new model of computation.

Acknowledgments

Appendix A. Dual Numbers

The theory of dual numbers was developed by Clifford. A dual number is a number of the following form: $\boldsymbol{a + b\varepsilon}$, where a and b are real numbers and $\varepsilon^2 = 0$.

Addition of dual numbers is defined as

$$(a + b\varepsilon) + (c + d\varepsilon) = (a + c) + (b + d)\varepsilon.$$

Multiplication of dual numbers is defined as

$$(a + b\varepsilon) \times (c + d\varepsilon) = (ac) + (ad + bc)\varepsilon.$$

It can be proven that

$$(a + b\varepsilon)^n = a^n + nba^{n-1}\varepsilon.$$

As well, it can be proven that there is an algebraic isomorphism between dual numbers and Bridge numbers: $\boldsymbol{a + b\varepsilon \Longleftrightarrow b\overline{0}\perp a}$.

The difference in our theory, which expands the Blow-up process so that the zero point is extended by distinguishing between the various multiplications of zero and creating a zero-axis. We also describe a completely new soft coordinate system with a twist, that includes the expansion of soft functions and their presentation on Soft curves.

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