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RESEARCH ARTICLES

Axioms of Soft Logic*

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Abstract—In this paper, we develop the foundation of a new mathematical language, which we term "Soft Logic". This language enables us to present an extension of the number 0 from a singular point to a continuous line. We create a distinction between -0 and +0 and generate a new type of numbers, which we call 'Bridge Numbers' (BN):

$$\boldsymbol{a\overline{0}\perp b\overline{1}},$$

where a, b are real numbers, "a" is the value on the $\overline{\mathbf{0}}$ axis, and "b" is the value on the $\overline{\mathbf{1}}$ axis. We proceed by defining arithmetic and algebraic operations on the Bridge Numbers, investigate their properties, and conclude by defining goals for further research. In the Attachment, we continue comparing our results with existing mathematical work on Infinitesimals, Dual numbers, and Non-standard analysis. The research is a part of "Digital living 2030" project with Stanford University.

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Key words: *soft logic, infinitesimal, nonstandard analysis, dual numbers, p-adic numbers.*

1. INTRODUCTION

1.1. Research Motivation and Direction

One of our motivations for this research is inspired by Marcelo Dascal [2], who wrote about the great mathematician and philosopher Gottfried Wilhelm Leibniz. As a young researcher, Leibniz aspired to discover and develop a mathematical language, which will demonstrate a softer logic that will overcome the limitations of the dichotomy of truth and false.

According to Dascal, language is a tool for thinking, but it also influences thinking. On many occasions Leibniz wrote that the logic containing only two states (1 and 0) is insufficient to grasp the full meaning of human reasoning. He did not develop this idea any further.

Another project of interest to Leibniz was to develop the world of the **Infinitesimals**. He saw them as ideal numbers, which might be infinitely small. The standard way to deal with infinitesimals in calculus is by using epsilon-delta procedures rather than the infinitesimals themselves. We come near the notion of infinitesimals in this paper by the invention of the zero axis.

In that context we can mention infinitesimals in other non-Archimedean fields, such as p-adic numbers. This interpretation was used in particular in p-adic probability theory by Andrei Khrennikov [3, 4].

Abraham Robinson [9] developed the theory of **Nonstandard Analysis** in 1960, which extended the real line $\mathbb{R}$ to the hyperreal $\mathbb{R}^*$ so that it includes infinitesimal numbers. In this theory, a positive infinitesimal is a number δ that satisfies $\frac{1}{n} > \delta > 0$ for every natural number n. Robinson uses the compactness theorem of logic to prove the existence of a universe with infinitesimals. This universe presentation is not constructive, and uses a weak form of the Zorn lemma (maximal ideal theorem). In this work, we present a constructive model for a continuum set of some special entities, which are close to infinitesimals and are called by us 'soft zeros', without using the Zorn lemma [5–8].

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Dual numbers were published in 1873 by William Clifford [1]. These numbers have the following form: $a + b\varepsilon$, where a, b are real numbers and $\varepsilon^2 = 0$. Dual numbers were used at the beginning of the twentieth century by Eduard Study [10] to represent dual angles. The set of dual numbers is isomorphic to the quotient of the polynomial ring $R[X]$ with the ideal generated by the polynomial X^2:

$$\frac{R[X]}{<X^2>}\ .$$

In this work we define a mathematical language that allows the creation of a geometrical bridge between internal and external knowledge (each inherently follows a different logic), and we call it Soft Logic. This language uses a set of some special tools, which we define and call 'Bridge Numbers' (BN).

1.2. Organization of the Paper

Section 2 presents three axioms of Soft Logic: Distinction, Addition, and Nullity.

Section 3 develops the positive soft number coordinate system, which represents a new system of numbers. This system makes a distinction between -0 and +0. It leads to the zero-line extension.

Section 4 presents the definition of Bridge Numbers, which connect the zero line and the real line of the coordinate system. Then, it presents the algebra of Bridge Numbers.

Section 5 ends the paper by suggesting further research and concluding remarks.

2. AXIOMS OF SOFT LOGIC

According to traditional mathematics, the expression 0/0 is undefined, although in fact any real number could represent this expression, since $a * 0 = 0$ for all real numbers a. This observation opens a new scope and area for investigation, which is a part of what we call "Soft Logic". Namely, we assume an existence of a continuum of multiples $a\bar{0}$ where a is any real number and by $\bar{0}$ we symbolize some special object that may be called, as well as any of his multiples, soft zero.

We denote by $\bar{\mathbf{1}}$ the real number 1 when all other real numbers are conceived as its multiples.

As well, we introduce the term of the absolute zero (bolded):

$$\mathbf{0} = 0\bar{0}.$$

Let a, b be any real numbers and $a\bar{0}, b\bar{0}$ two corresponding zeros defined above.

Axiom 1 (Distinction): If $a \neq b$ then $a\bar{0} \neq b\bar{0}$.

In Soft Logic, we extend the zero from a point to a line. This creates a distinction between different multiples of $\bar{\mathbf{0}}$.

Definition (Order): If $a < b$ then $a\bar{0} < b\bar{0}$.

Axiom 2 (Addition): $a\bar{0} + b\bar{0} = (a + b)\bar{0}$.

Under the assumption that the multiples of $\bar{0}$ are located on a straight line, we can define addition of multiples of $\bar{0}$ as the addition of their corresponding real multipliers. The $\bar{\mathbf{1}}$ axis behaves regularly:

$$a\bar{1} + b\bar{1} = (a + b)\bar{1}.$$

Axiom 3 (Nullity): $a\bar{0} * b\bar{0} = 0$.

Numbers on the zero axis "collapse" under multiplication. Addition has a significance and meaning, but multiplication does not make any distinction whatsoever.

On the other hand:

$$a\bar{1} * b\bar{1} = ab\bar{1},$$

$$a\bar{0} * b\bar{1} = ab\bar{0},$$

$$b\bar{1} * a\bar{0} = ba\bar{0}.$$

3. THE POSITIVE SOFT NUMBER SYSTEM

We suggest a new coordinate axis, as presented in the following figure (**Figure 1**): It starts from 0 to 1 horizontally and then it turns 90^0 from 1 to infinity.

Lemma 1: There exists a one to one correspondence between the segment (0, 1] and the segment [1,∞).

Proof: Consider the function $f(x) = \frac{1}{x}$. This function creates a one to one correspondence between $(0, 1]$ and $[1, \infty)$. ■

The **Figure 1** contains example of a line between $x = 1/2$ and $x = 2$.

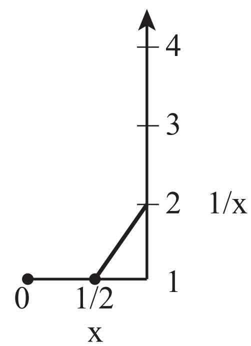

Fig.1

Also, one can draw such a line from any nonzero real x to $1/x$ (**Figure 2**).

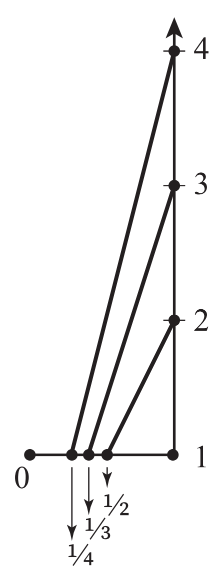

Fig.2

Lemma 2: All lines that connect x to $1/x$ (for all non-zero real x), intersect at a single point.

Proof: Let us observe the following drawing with a line, which **connects** x with $1/x$ (**Figure 3**).

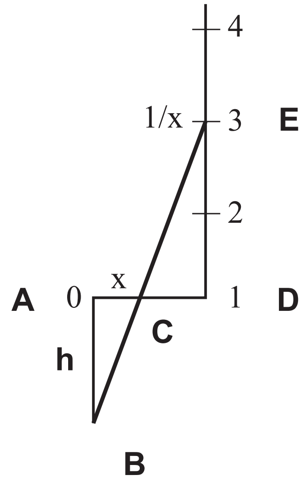

Fig.3

$\Delta ABC \approx \Delta CDE$ as triangles with equal angles. Consequently,

$$\frac{AC}{AB} = \frac{CD}{ED}.$$

Therefore,

$$\frac{x}{\mathrm{h}} = \frac{1-x}{\frac{1}{x}-1}.$$

Hence

$$h = 1.$$

This means that all lines that connect x to 1/x intersect at the same point, which is located one unit below the 0. ■

The intersection point denotes the beginning of the soft logic coordinate system. We call this point "the absolute zero". The distance from the absolute zero to $+\overline{\mathbf{0}}$ is 1. We suggest extending this new coordinate axis symmetrically, to the negative numbers (**Figure 4**).

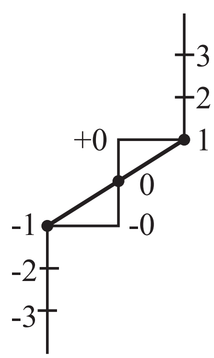

Fig.4

We now have instead of one zero 0 three zeroes. One zero is opposite to the number -1, and is not identical to the zero opposite to the number +1. Hence, we suggest denoting the two different "zeroes" as $-\bar{\mathbf{0}}$ and $+\bar{\mathbf{0}}$. Furthermore, the discovery of the ***zero line*** enables us to find locations of the multiples of the soft zero **(Figure 5).**

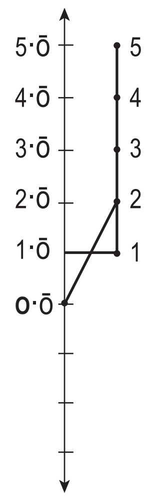

Fig.5

The zero line is developed according to the three axioms presented above. The following drawing presents the extended new coordinate system for positive and negative numbers with the multiplication of the soft zero **(Figure 6).**

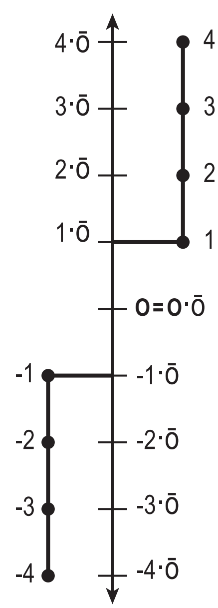

Fig.6

4. BRIDGE NUMBERS

Following the concept of the zero line, we define a new type of numbers, the Bridge numbers. They have two components: multiples of $\bar{0}$ and multiples of $\bar{1}$.

A Bridge number is a number of the following form, where a and b are real numbers:

$$a\bar{0}\perp b\bar{1}.$$

Note the new symbol $\perp$, which represents the bridging operation (it is not the same as a regular plus, and the whole expression has to be read as '$a\bar{0}$ bridge $b\bar{1}$').

The set of all Bridge numbers is $\mathrm{BN} = \{a\bar{0}\perp b\bar{1} : a, b \in \mathbb{R}\}$.

For the sake of brevity, we remove the presence of $\bar{1}$ and from here on present the Bridge numbers in the form:

$$a\bar{0}\perp b.$$

We do not yet place these numbers in the soft coordinate systems.

4.1. BN is a Group

We define addition of Bridge numbers according to Axiom 2 (Addition) of Soft Logic:

$$\left(a\bar{0}\perp b\right) + \left(c\bar{0}\perp d\right) = \left(a\bar{0} + c\bar{0}\right)\perp(b+d) = (a+c)\,\bar{0}\perp(b+d)\,.$$

The neutral element of addition in BN is $\left(0\bar{0}\perp 0\right)$ because

$$\left(a\bar{0}\perp b\right) + \left(0\bar{0}\perp 0\right) = a\bar{0}\perp b.$$

The inverse element of $\left(a\bar{0}\perp b\right)$ in BN is $\left(-a\bar{0}\perp(-b)\right)$ because

$$\left(a\bar{0}\perp b\right) + \left(-a\bar{0}\perp(-b)\right) = 0\bar{0}\perp 0.$$

The associative law is satisfied in BN:

$$\left(a\bar{0}\perp b\right) + \left(\left(c\bar{0}\perp d\right) + (e\bar{0}\perp f)\right)$$

$$= \left(a\bar{0}\perp b\right) + \left((c+e)\,\bar{0}\perp(d+f)\right) = (a+c+e)\,\bar{0}\perp(b+d+f)$$

$$= \left((a\bar{0}\perp b\right) + \left(\mathrm{c}\bar{0}\perp d)\right) + (e\bar{0}\perp f).$$

The addition is also commutative:

$$\left(a\bar{0}\perp b\right) + \left(\mathrm{c}\bar{0}\perp d\right) = \left(c\bar{0}{+}d\right)\perp\left(a\bar{0}\perp b\right) = (a+c)\,\bar{0}\perp(b+d)\,.$$

Thus, BN is a commutative group under the operation of addition.

4.2. BN is a Ring

We define multiplication according to Axioms 2 and 3 of Soft Logic:

$$\left(a\bar{0}\perp b\right) * \left(c\bar{0}\perp d\right) = \left(ad\bar{0}{+}bc\bar{0}\right)\perp bd$$

$$= (ad+bc)\,\bar{0}\perp bd.$$

Under this definition, the multiplication in BN satisfies the distributive law:

$$\left(a\bar{0}\perp b\right) * \left(\left(c\bar{0}\perp d\right) + (e\bar{0}\perp f)\right)$$

$$= \left(a\bar{0}\perp b\right) * \left((c+e)\,\bar{0}\perp(d+f)\right) = (ad+af+bc+be)\,\bar{0}\perp(bd+bf)$$

$$= \left(a\bar{0}\perp b\right) * \left(c\bar{0}\perp d\right) + \left(a\bar{0}\perp b\right) * (e\bar{0}\perp f).$$

Additionally, the multiplication in BN satisfies the associativity law and has a neutral element $0\cdot\bar{0}\perp 1$ (clearly).

Therefore, BN is a ring under addition and multiplication.

4.3. BN is Almost a Field

Lemma 3: If $b \neq 0$, then

$$\frac{1}{a\overline{0}\bot b} = -\frac{a}{b^2}\overline{0}\bot\frac{1}{b}.$$

Proof: The equality

$$\frac{1}{a\overline{0}\bot b} = x\overline{0}\bot y$$

means that

$$\left(a\overline{0}\bot b\right) * (x\overline{0}\bot y) = (ay + bx)\,\overline{0}\bot by = 1,$$

$$by = 1,$$

$$ay + bx = 0,$$

$$y = \frac{1}{b},$$

$$x = -\frac{a}{b^2}.$$

Therefore, BN is almost a field, since all elements (excluding those on the zero line) have an inverse. ■

Corollary: Numbers on the $\overline{0}$ axis have no inverse.

4.4. Power of Bridge Numbers

We can calculate the square of a Bridge number:

$$\left(a\overline{0}\bot b\right)^2 = (a\overline{0} * a\overline{0} + 2ab\overline{0})\bot b^2 = 2ab\overline{0}\bot b^2$$

$$\left(a\overline{0}\bot b\right)^2 = 2ab\overline{0}\bot b^2.$$

Calculation of powers $(a \cdot \overline{0}\bot b)^n$ where n is a positive integer is as given by the following preposition:

Lemma 4: $\left(a\overline{0}\bot b\right)^n = nab^{n-1}\overline{0}\bot b^n$.

Proof:

$$\left(a\overline{0}\bot b\right)^n = \left(a\overline{0}\bot b\right) * \ldots * \left(a\overline{0}\bot b\right).$$

In these multiplications, there are n terms of $\left(a\overline{0}\bot b\right)$. When we calculate the product, each time, we need to choose between one of the terms, $\left(a\overline{0}\right)$ *or* (b). Any multiplication of two $\left(a\overline{0}\right)$ gives 0 and thus vanishes. Therefore $\left(a\overline{0}\right)$ has to be multiplied only with (n-1) numbers b. There are n possibilities to choose an element $\left(a\overline{0}\right)$ that we multiply by $\left(b^{n-1}\right)$. Thus, the component $nab^{n-1}\overline{0}$ before the bridge sign is received. To get the one after this sign, all n real numbers b are to be multiplied which gives b^n.

Therefore,

$$\left(a\overline{0}\bot b\right)^n = nab^{n-1}\overline{0}\bot b^n.$$

As well, the statement can be proved by induction for $n \geq 2$. ■

4.5. Root of Bridge Numbers

We can calculate the square root of a Bridge number:

$$\sqrt{a\overline{0}\bot b} = x\overline{0}\bot y.$$

Lemma 5: If $b > 0$, then $\sqrt{a\overline{0}\bot b} = \pm\frac{a}{2\sqrt{b}}\overline{0}\bot \pm \sqrt{b}$.

Proof:

$$\left(x\overline{0}\bot y\right)^2 = 2xy\overline{0}\bot y^2,$$

$$2xy = a,$$

$$y^2 = b,$$

$$y = \pm\sqrt{b},$$

$$x = \pm\frac{a}{2\sqrt{b}}.$$

For $b = 0$, $a \neq 0$ there is no solution (one cannot multiply a zero-axis number by itself and receive a zero-axis number, except when $a = 0$).

Therefore, there are at most two square roots for every Bridge number.

We check that by the following calculation:

$$\left(\frac{a}{2\sqrt{b}}\overline{0}\bot\sqrt{b}\right)^2 = 2\sqrt{b}\frac{a}{2\sqrt{b}}\overline{0}\bot\left(\sqrt{b}\right)^2 = a\overline{0}\bot b,$$

$$\left(\frac{-a}{2\sqrt{b}}\overline{0}\bot - \sqrt{b}\right)^2 = 2(-\sqrt{b})\frac{-a}{2\sqrt{b}}\overline{0}\bot\left(-\sqrt{b}\right)^2 = a\overline{0}\bot b.$$

■

Lemma 6: The general nth root of a Bridge number is: For $b > 0$ or odd n and non-zero b, $\sqrt[n]{a\overline{0}\bot b} = \frac{a}{nb^{\frac{n-1}{n}}}\overline{0}\bot \sqrt[n]{b}$, and for even n, also: $\sqrt[n]{a\overline{0}\bot b} = -\frac{a}{nb^{\frac{n-1}{n}}}\overline{0}\bot - \sqrt[n]{b}$. For $b = 0$ and $a \neq 0$, there is no solution.

Proof:

$$\left(x\overline{0}\bot y\right)^n = nxy^{n-1}\overline{0}\bot y^n,$$

$$a\overline{0} + b = nxy^{n-1}\overline{0}\bot y^n,$$

$$y^n = b,$$

$$nxy^{n-1} = a.$$

For odd n, (assuming b nonzero)

$$y = \sqrt[n]{b},$$

$$x = \frac{a}{ny^{n-1}} = \frac{a}{nb^{\frac{n-1}{n}}}.$$

For $b = 0$ and $a \neq 0$, there is no solution.

We can verify this by the following calculation:

$$\left(\frac{a}{nb^{\frac{n-1}{n}}}\overline{0}\bot\sqrt[n]{b}\right)^n = n\left(\sqrt[n]{b}\right)^{n-1}\frac{a}{nb^{\frac{n-1}{n}}}\overline{0}\bot\left(\sqrt[n]{b}\right)^n = a\overline{0}\bot b.$$

For even n and $b>0$, the calculation similar to one above gives:

$$y = \pm \sqrt[n]{b},$$

$$x = \frac{a}{ny^{n-1}} = \frac{\pm a}{nb^{\frac{n-1}{n}}}.$$

We can verify the additional minus-formula by the following calculation:

$$\left(\frac{-a}{nb^{\frac{n-1}{n}}}\bar{0}\perp - \sqrt[n]{b}\right)^n = n\left(-\sqrt[n]{b}\right)^{n-1}\frac{-a}{nb^{\frac{n-1}{n}}}\bar{0}\perp\left(-\sqrt[n]{b}\right)^n = a\bar{0}\perp b.$$

■

We showed the richness of the algebra of BN: It is a commutative group, a ring, and almost a field, and the powers and the roots have a particular nature.

5. CONCLUSIONS AND FURTHER RESEARCH

In this paper, we present how to extend the number zero from a point to a continuous line of the multiples of the soft-zero. We developed a new coordinate system with the 0-axis and the 1-axis. We defined Bridge numbers that create an algebraic structure, which is almost a field.

By this, we opened a new area for investigation, and we plan to research and develop it further. For example, the logic of the zero line (Soft Logic) can be a mathematical model to describe the human inner experience.

In further research, we plan to investigate and add a non-commutative axiom of the Bridge numbers, and thus to create soft numbers. We plan to define the continuous representation of soft numbers in the band between the 0-axis and the 1-axis. We want to extend real function to function over the set of soft numbers.

To conclude with the future vision, we mention that there is a discussion among scientists that behind the space-time dimension there is another world, which is the world of information. In further research, we want to investigate the zero-axis as a possible origin of this world.

6. APPENDICES

Related work

We present three topics, and explain their connections, similarities and dissimilarities, to Soft Logic: Nonstandard analysis, Dual numbers and the Ring of polynomials.

6.1. Nonstandard Analysis

Nonstandard analysis was developed by Abraham Robinson, in 1960. Shortly, this work extends the set of real numbers $\mathbb{R}$ to the set of hyperreals $\mathbb{R}^*$, which includes infinitesimal numbers. While Robinson relied on the compactness theorem of logic, others (like Luxemburg, Keislar) preferred a more direct construction of "Ultrapowers". By them $\mathbb{R}^* = \frac{\mathbb{R}^\omega}{M}$, where M is an ideal of all infinite sequences that are equivalent to 0, under a chosen Ultrafilter F. A number $\delta \epsilon \mathbb{R}^*$is called infinitesimal if $-\varepsilon < \delta < \varepsilon$, for every real positive number .

In the ultrapower $\mathbb{R}^\omega$, infinite sequences describe the elements of $\mathbb{R}^*$. The standard real number c is identified with the sequence $(c, c, c, \dots) \equiv c$.

Two infinite sequences, $a = (a_1, a_2, a_3 \dots)$, $b = (b_1, b_2, b_3 \dots)$ are equal iff the set of indices that have equal elements, i.e. $a_i = b_i$, is a big set.

A big set is a member of an Ultrafilter F, which in turn is a non-empty set of subsets of the natural numbers set N that satisfies the following conditions:

$$1.\ For\ all\ A, B,\ if\ A \in F\ and\ \ A \subset B,\ then\ B \in F.$$

$$2.\ For\ all\ A_1, A_2 \in F \Longrightarrow A_1 \cap A_2 \in F.$$

$$3.\ For\ all\ A \in P(N),\ either\ A\ \in F\ or\ A^c \in F.$$

In Robinson's model of $\mathbb{R}^*$ the sequence

$\delta = \left(1\ .\frac{1}{2}, \frac{1}{3}, \frac{1}{4},\ \ldots .\frac{1}{n},\ \ldots .\right)$ is an infinitesimal number because for any natural m there is $0 < \delta < \frac{1}{m}$.

The disadvantage of Robinson's construction of $\mathbb{R}^*$ is that it uses non-constructive tools. In soft logic, we built a model for the soft zeros that are in some way close to infinitesimals using the zero line, without the Zorn Lemma or compactness theorem. In the following, we show how to represent Bridge numbers by Nonstandard analysis.

How to represent Bridge numbers by using a model from Nonstandard analysis.

In $\mathbb{R}^*$, there exists an infinitesimal $y > 0$ but there is a difference between y and $\overline{0}$ since $\left(\overline{0}\right)^2 = 0$ but $y^2 \neq 0$. It is possible to treat y^2 as very small and to neglect it, and by that to connect it more strongly to $\overline{0}$. In order to do this, we define when a number δ is "very small" in relation to y.

Definition: An infinitesimal δ is neglected in relation to infinitesimal y iff $\frac{\delta}{y}$ is infinitesimal.

We denote by I_y the set of all infinitesimals, which are neglected in relation to $i$nfinitesimal y.

Lemma: If y is an infinitesimal, then $y^2 \in I_y$.

Proof: Since $\frac{y^2}{y} = y$ and y is an infinitesimal,the statement is true.

For a given infinitesimal y we define the following set:

$$K = \{ay + b + \delta : a, b \in \mathbb{R}, \delta \in I_y\}.$$

This set is a subset of $\mathbb{R}^*$ closed under addition and multiplication.

We can define mapping D of K on BN: K->BN, where

$D\,(ay + b + \delta) = a\overline{0}\perp b$ for any $\delta \in I_y$.

It can be shown that for all x, z in K we get $D(x + z) = D(x) + D(z)$ and $D(x{*}z) = D(x){*}D(z)$. Therefore the mapping D is homomorphism. As well, it has following properties:

(i) $D(y) = \overline{0}$.

(ii) For all x in K, we have: $D(x) = 0$ iff x is in I_y. This proves that $D(y){*}D(y) = 0$.

(iii) For all real x, $D(x) = x$.

The analogy of K to BN is as follows:

Using D, numbers of the form $ay{+}b{+}\delta$ are mapped into $a\overline{0}\perp b$ (numbers in I_y are mapped to 0, and y is mapped to $\overline{0}$). As we have that $y^2 \in I_y$, it means that y^2 becomes 0, after the mapping. ■

6.2. Dual Numbers

The theory of Dual numbers was developed by Clifford. A dual number is a number of the following form: $\boldsymbol{a + b\varepsilon}$**,** where a and b are real numbers and $\varepsilon^2 = 0$.

Addition of dual numbers is defined as

$$(a + b\varepsilon) + (c + d\varepsilon) = (a + c) + (b + d)\varepsilon.$$

Multiplication of dual numbers is defined as

$$(a + b\varepsilon) \times (c + d\varepsilon) = (ac) + (ad + bc)\varepsilon.$$

It can be proven that

$$(a + b\varepsilon)^n = a^n + nba^{n-1}\varepsilon.$$

As well, it can be proven that there is an algebraic isomorphism between dual numbers and Bridge numbers:

$$\boldsymbol{a + b\varepsilon \Longleftrightarrow b\overline{0}\perp a}.$$

6.3. The Ring of Polynomials

The ring of polynomials R[X] is the set of all finite polynomials over the real numbers under addition and multiplication:

$$R\,[X] = \{a_0 + a_1 X + a_2 X^2 + \ldots a_n X^n \ : \ X, \ a_i \in \mathrm{R}\}.$$

The algebra of dual numbers is equivalent to the algebra of quotient ring:

$$\frac{R\,[X]}{< X^2 >} = \{a + bX : a, b, \ X \in \mathbb{R}\}.$$

The isomorphism between Bridge numbers and the quotient ring of polynomials is given by

$$\boldsymbol{a + bX \Longleftrightarrow b\bar{0} \perp a}.$$

The main difference between the sets of Bridge and dual numbers and the quotient ring of polynomials is in their geometric interpretation, which will be presented in the next paper. Here we would like to note the following:

Soft logic creates a new coordinate system that makes a bridge between objectivity and subjectivity.

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REFERENCES

1. W. Clifford, “Preliminary sketch of Bi-quaternions,” Proc. London Math. Soc. **4**, 381–95 (1873).
2. M. Dascal (Ed.), *Leibniz: What Kind of Rationalist?* Logic, Epistemology, and the Unity of Science, **13** (Springer, Netherlands, 2008).
3. A. Khrennikov, *Information Dynamics in Cognitive, Psychological, Social, and Anomalous Phenomena*, Theories of Physics (Kluwer, Dordreht, 2004).
4. A. Khrennikov, “Interpretations of probability and their p-adic extension,” Theory Probab. Appl. **46** (2), 312–325 (2001).
5. M. Klein and O. Maimon, “The mathematics of soft logic,” IOP Second International Conference on Mechanical Engineering and Automation Science (Singapore, 13–15 October 2016).
6. M. Klein and O. Maimon, “Developing soft logic in kindergarten,” to be published at Literature and Arts Review, ISSN 2517-9446, USA (2019).
7. M. Klein and O. Maimon, *Soft Logic and Soft Numbers*, Pragmatics & Cognition (John Benjamins Publishing Company, 2016).
8. Y. Landsberg, *A Course in Mind-Bending And/Vs. Mind-Binding*, Personal Notes (2018).
9. A. Robinson, *Non-Standard Analysis* (Princeton University Press, 1974).
10. E. Study, *Geometrie der Dynamen* (Leibzig, 1903).